MATHEMATICS

ON SOME SPECIAL THETA FUNCTIONS

BY

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1. *Preliminaries*

In this paper we shall use the usual notation $\{\Gamma, -r, v\}$ for the class of modular forms of dimension $-r$ for the group Γ, with multiplier system v.

$\Gamma[1]$ is the modular group, i.e. the group of integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc=1$. We use the same notation for the associated group of transformations $\tau \to \frac{a\tau+b}{c\tau+d}$.

We write:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$\Gamma[N]$ is the subgroup of $\Gamma[1]$ defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm I \pmod N$.

$\Gamma_0[N]$ and $\Gamma^0[N]$ are the subgroups of $\Gamma[1]$ defined by $c \equiv 0 \pmod N$ and $b \equiv 0 \pmod N$ respectively.

Γ_ϑ is the group generated by U^2 and T.

The functions η, ϑ_{gh} and G_k are defined as follows:

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1-e^{2\pi i n\tau}) \qquad (\mathrm{Im}\ \tau > 0).$$

$$\vartheta_{gh}(\tau) = \sum_{n=-\infty}^{+\infty} (-1)^{nh} e^{\pi i \tau (n+{}^1\!/_2 g)^2} \qquad (\mathrm{Im}\ \tau > 0, g \text{ and } h \text{ integers}).$$

$$G_k(\tau) = \frac{1}{2\zeta(k)} \sum_{m_1, m_2}{}' (m_1\tau + m_2)^{-k} \qquad (\mathrm{Im}\ \tau > 0, k \text{ even}).$$

If S is the matrix (s_{ij}) we shall write the quadratic form $\sum_{i,j=1}^{n} s_{ij}x_i x_j$ as $x'Sx$.

2. *The theta functions*

It is well known that the theta functions ϑ_{00}, ϑ_{01} and ϑ_{10} are modular forms for the groups Γ_ϑ, $\Gamma^0[2]$ and $\Gamma_0[2]$ respectively. These groups are subgroups of index 3 of the full modular group $\Gamma[1]$. From these functions one can form modular forms for the group $\Gamma[1]$. A well known example is the fact that $\vartheta_{01}^{4k} + \vartheta_{10}^{4k} + (-1)^k \vartheta_{00}^{4k} = F_k$ where $F_k \in \{\Gamma[1], -2k, v\}$. Here v is determined by $v(U) = (-1)^k$.

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We shall now give a more general definition of theta functions and try to construct modular forms for $\Gamma[1]$ with these. We consider an integral positive definite quadratic form $x'Sx$ in n variables and two arbitrary vectors a and b and consider the function

$$\sum_{x\equiv a(\mathrm{mod}\,1)} e^{\pi i\tau x'Sx}\, e^{2\pi i x'Sb} \qquad (\mathrm{Im}\ \tau>0).$$

The factor $e^{\pi i x'Sx}$ enters into the sum when we consider the transformation $\tau\to\tau+1$. We introduce a vector w with the property

$$x'Sx \equiv 2x'Sw \pmod 2$$

for all integral x. It is easily seen that this is possible and that we can take $w=0$ if $x'Sx$ is an even form.

We shall require that S has determinant 1. Then the transformation $\tau\to\frac{-1}{\tau}$ maps the function into another function of the same type multiplied by a factor which depends on a and b. We now change our definition so that these transformation formulae no longer depend on a and b. In this way we come to the definition

$$\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right) = e^{-\pi i\{a'Sb+2a'Sw+w'Sw\}} \sum_{x\equiv a+w(\mathrm{mod}\,1)} e^{\pi i\tau x'Sx}\, e^{2\pi i x'S(b+w)}. \tag{2.1}$$

The transformation formulae are

$$\vartheta\left(\tau\middle|\begin{matrix}a+g\\b+h\end{matrix}\right) = e^{\pi i\{a'Sh-b'Sg\}}\,(-1)^{g'Sg+h'Sh+g'Sh}\,\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right) \tag{2.2}$$

for integral vectors g and h.

$$\vartheta\left(\tau\middle|\begin{matrix}-a\\-b\end{matrix}\right) = e^{4\pi i w'Sw}\,\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right), \tag{2.3}$$

$$\vartheta\left(\tau+1\middle|\begin{matrix}a\\b\end{matrix}\right) = e^{\pi i w'Sw}\,\vartheta\left(\tau\middle|\begin{matrix}a\\a+b\end{matrix}\right), \tag{2.4}$$

$$\vartheta\left(\frac{-1}{\tau}\middle|\begin{matrix}a\\b\end{matrix}\right) = \tau^{\frac{n}{2}}\, e^{-\frac{\pi i n}{4}}\, e^{-2\pi i w'Sw}\,\vartheta\left(\tau\middle|\begin{matrix}b\\-a\end{matrix}\right). \tag{2.5}$$

Using $(UT)^3=-I$ we find from these relations:

$$\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right) = e^{\pi i\left\{\frac{n}{4}-w'Sw\right\}}\,\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right).$$

As S is positive definite we have $\vartheta\left(\tau\middle|\begin{matrix}-w\\-w\end{matrix}\right)\neq 0$ and hence $\frac{n}{4}\equiv w'Sw \pmod 2$. If S is an even form we find the condition $n\equiv 0 \pmod 8$ (cf. [1]).

In some cases it is easier to work with modular functions than modular forms. We therefore define

$$\varphi\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right) = \vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right)\cdot\eta^{-n}(\tau). \tag{2.6}$$

We have:

$$\varphi\left(\tau+1\middle|\begin{matrix}a\\b\end{matrix}\right)=e^{\frac{\pi i n}{6}}\varphi\left(\tau\middle|\begin{matrix}a\\a+b\end{matrix}\right), \tag{2.7}$$

$$\varphi\left(\frac{-1}{\tau}\middle|\begin{matrix}a\\b\end{matrix}\right)=e^{-\frac{\pi i n}{2}}\varphi\left(\tau\middle|\begin{matrix}b\\-a\end{matrix}\right). \tag{2.8}$$

From (2.7) and (2.8) we find for $L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma[1]$ the transformation formula

$$\varphi\left(L\tau\middle|\begin{matrix}a\\b\end{matrix}\right)=\varepsilon(L)\cdot\varphi\left(\tau\middle|\begin{matrix}\alpha a+\gamma b\\ \beta a+\delta b\end{matrix}\right). \tag{2.9}$$

Here $\varepsilon(L)$ is a character of the group $\Gamma[1]$. If we write $\xi=e^{\frac{\pi i n}{6}}$ we have (cf. [2])

$$\varepsilon(L)=\begin{cases}\xi^{(\alpha+\delta)\gamma-\beta\delta(\gamma^2-1)-3\gamma} & \text{if } \gamma\equiv 1 \pmod 2,\\ \xi^{(\alpha+\delta)\gamma-\beta\delta(\gamma^2-1)+3\delta-3-3\gamma\delta} & \text{if } \delta\equiv 1 \pmod 2.\end{cases} \tag{2.10}$$

We shall now restrict ourselves to the case where there is an integer N such that Na and Nb are integral vectors. Then, by (2.9) and (2.2), we have, if $L\equiv\pm I \pmod N$):

$$\varphi\left(L\tau\middle|\begin{matrix}a\\b\end{matrix}\right)=\varepsilon(L)\varrho(L,a,b)\,\varphi\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right). \tag{2.11}$$

Hence φ is a modular function of $\{\Gamma[N], 0, \varepsilon\varrho\}$.

Here $\varrho(L, a, b)$ is a root of unity which depends on L, a and b in a complicated way. We introduce the following notation:

$$A=N^2a'Sa,\quad B=N^2b'Sb,\quad\text{and}\quad C=2N^2a'Sb.$$

We take $L\equiv I \pmod N$, and write:

$$L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+N\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}.$$

Then we find:

$$\varrho=e^{\frac{\pi i}{N}(\beta'A+\delta'C-\gamma'B)}\,(-1)^{(\alpha'+\alpha'\beta'+\beta')A+(\gamma'+\gamma'\delta'+\delta')B}. \tag{2.12}$$

If N is odd this reduces to

$$\varrho=e^{\pi i\frac{N+1}{N}(\beta'A-\gamma'B)}\,e^{\pi i\frac{\delta'C}{N}}.$$

3. *Relations for the theta functions*

We wish to find linear combinations of the functions φ which are a modular function for $\Gamma[1]$. It is clear that we can restrict the a and b to vectors of a lattice generated by two given vectors a_0 and b_0. Now if the function of $\Gamma[1]$ is not identically zero it generally must have the form $\varphi|T[K, \Lambda]$ where $T[K, \Lambda]$ is a general Hecke-operator introduced by WOHLFAHRT ([3]). From the theory of these we see that a necessary condition is that there is a character of $\Gamma[1]$ which is equal to $\varepsilon\varrho$ on the subgroup $\Gamma[N]$. In this way we find restrictions for the vectors a and b

that are to be used. We shall distinguish different values of N and give some necessary conditions for the existence of linear combinations which belong to $\Gamma[1]$.

The value of the character for U^N must be $\varepsilon(U^N)\,\varrho(U^N, a, b)$. This gives us the condition

$$nN^2 + 6A(N+1) \equiv 0 \bmod \{2N\cdot(6, N)\}.$$

We distinguish

1) $N \equiv 1 \pmod 2$.

Then the above condition is equivalent with $n \equiv 0 \pmod 2$ and $A \equiv 0 \pmod N$. Using $\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix}$ and $\begin{pmatrix}1+N & N\\ N & 1-N\end{pmatrix}$ we find in the same way: $B \equiv 0 \pmod N$ and $C \equiv 0 \pmod{2N}$. In this case $\varrho = 1$.

2) $N \equiv 0 \pmod 4$ or $N \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 2$.

In the same way as above we find $A \equiv B \equiv C \equiv 0 \pmod{2N}$ and $\varrho = 1$.

3) $N \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 2$.

We find $A \equiv B \equiv C \equiv N \pmod{2N}$ and $\varrho = (-1)^{\beta' + \delta' - \gamma'}$.

In some cases there are different characters of $\Gamma[1]$ which are equal to $\varepsilon\varrho$ on $\Gamma[N]$. To get linear combinations of the functions φ which are not identically zero and belong to $\Gamma[1]$ we must consider a function φ and the largest subgroup of $\Gamma[1]$ for which it is a modular function and then find a character of $\Gamma[1]$ which has the same value on this subgroup. Even then we can sometimes find two or more such characters. In 4 we shall give an example of this case.

4. *Examples*

We shall give some examples of the relations one can find for the functions φ. We consider three different cases, namely

1) $a = b = 0$, 2) $a = 0$, b arbitrary, and 3) a and b linearly independent.

4.1. $a = b = 0$. In this case $\varphi(\tau) = \varphi\left(\tau\middle|\begin{matrix}0\\0\end{matrix}\right)$ is already a modular function for the group $\Gamma[1]$. From (2.3) we see that $\vartheta\left(\tau\middle|\begin{matrix}0\\0\end{matrix}\right) = (-1)^n\vartheta\left(\tau\middle|\begin{matrix}0\\0\end{matrix}\right)$ and hence if $\varphi \neq 0$, n must be even. In fact if $n \leqslant 11$ we have $\vartheta\left(\tau\middle|\begin{matrix}0\\0\end{matrix}\right) = 0$, except if $n = 8$ and S is an even form.

For $n = 12$ and $x'Sx = 23x_1^2 + 2\sum_{i=2}^{12} x_1^2 + 10x_1x_2 + 2\sum_{i=2}^{11} x_i x_{i+1}$ we have

$$\vartheta\left(\tau\middle|\begin{matrix}0\\0\end{matrix}\right) = -\sum_{x \equiv (0,\frac12,0,\frac12,\ldots,0,\frac12) \bmod 1} e^{\pi i\tau x'Sx}\, e^{\pi i x_1} = 24\eta^{12}(\tau) \text{ (cf. [4])}.$$

4.2. $a = 0$. In this case the functions φ are modular functions for the group generated by U and $\Gamma[N]$. If $N = 2$ we have

$$\varphi\left(\tau\middle|\begin{matrix}0\\b\end{matrix}\right) + \varphi\left(\tau\middle|\begin{matrix}b\\0\end{matrix}\right) + \varphi\left(\tau\middle|\begin{matrix}b\\b\end{matrix}\right) = \Phi(\tau) \text{ with } \Phi \in \{\Gamma[1], 0, v\}.$$

Here a necessary condition is that $b'Sb$ is an integer. As example we take for S the unit matrix. If one of the components of b is 0 we have $\varphi = 0$. Therefore we can assume that all components of b are $\frac12$. Then $n \equiv 0 \pmod 4$.

After multiplying by η^{4k} the relation becomes

$$\vartheta_{01}^{4k}+\vartheta_{10}^{4k}+(-1)^k\vartheta_{00}^{4k}=F_k\varepsilon\{\Gamma[1],\,-2k,v\}. \tag{4.1}$$

We have: $F_1=0$; $F_2=2G_4$; $F_3=-48\eta^{12}$; $F_4=2G_8$.

This is a well known result.

If $N=3$ we have

$$\varphi\left(\tau\middle|\begin{matrix}0\\b\end{matrix}\right)+\varphi\left(\tau\middle|\begin{matrix}b\\0\end{matrix}\right)+\varphi\left(\tau\middle|\begin{matrix}b\\b\end{matrix}\right)+\varphi\left(\tau\middle|\begin{matrix}-b\\b\end{matrix}\right)=\Phi(\tau) \text{ with } \Phi\in\{\Gamma[1],0,v\}.$$

Here a necessary condition is that $b'Sb\equiv 0 \pmod 3$ and $n\equiv 0 \pmod 2$. If we take as S the unit matrix we again have $\varphi=0$ if one of the components of $b=0$. It is easily seen that we can therefore restrict ourselves to vectors b with all components equal to $\frac{1}{2}$. Then we have $n\equiv 0 \pmod 6$ and as

$$\sum_{x\equiv\frac{1}{2}(\mathrm{mod}\,1)} e^{\pi i\tau x^2}e^{2\pi i^5/_6 x}=-\sqrt{3}\,\eta(3\tau)$$

we find a relation which can be written as

$$\eta^{6k}|T(3)\in\{\Gamma[1],\,-3k,v\}, \tag{4.2}$$

where $T(3)$ is a generalized Hecke-operator (cf. [3]).

For $k=1$ we have $\eta^6|T(3)=0$, for $k=2$: $\eta^{12}(\tau)|T(3)=c\cdot\eta^{12}(\tau)$.

Finally we give an example with $N=5$, $n=2$. Let S be the unit matrix and $a=0$, $b=\dfrac{1}{5}(1,\,2)$. Then

$$\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right)=-i\Big\{\sum_{x\equiv\frac{1}{2}(1)}e^{\pi i\tau x^2}e^{2\pi i^7/_{10}x}\Big\}\Big\{\sum_{x\equiv\frac{1}{2}(1)}e^{\pi i\tau x^2}e^{2\pi i^9/_{10}x}\Big\}=$$

$$=-4i\left\{\sum_{n=0}^{\infty}e^{\frac{\pi i\tau(2n+1)^2}{4}}\cos\frac{3(2n+1)}{10}\pi\right\}\left\{\sum_{n=0}^{\infty}e^{\frac{\pi i\tau(2n+1)^2}{4}}\cos\frac{(2n+1)\pi}{10}\right\}$$

and $\varphi(\tau)=\dfrac{\vartheta\left(\tau\middle|\begin{smallmatrix}a\\b\end{smallmatrix}\right)}{\eta^2(\tau)}$.

$\varphi\in\{\Gamma_0[5],\,0,\,v\}$ where v is a character of $\Gamma[1]$. The relation is

$$\vartheta\left(\tau\middle|\begin{matrix}0\\b\end{matrix}\right)+\vartheta\left(\tau\middle|\begin{matrix}b\\b\end{matrix}\right)+\vartheta\left(\tau\middle|\begin{matrix}2b\\b\end{matrix}\right)+\vartheta\left(\tau\middle|\begin{matrix}3b\\b\end{matrix}\right)+\vartheta\left(\tau\middle|\begin{matrix}4b\\b\end{matrix}\right)+\vartheta\left(\tau\middle|\begin{matrix}b\\0\end{matrix}\right)=0.$$

4.3. a and b independent.

In all the examples S will be the unit matrix.

i) $N=2$, $n=3$, $a=(\frac{1}{2},\frac{1}{2},0)$, $b=(0,\frac{1}{2},\frac{1}{2})$.

$\vartheta\left(\tau\middle|\begin{smallmatrix}a\\b\end{smallmatrix}\right)=-\vartheta_{00}(\tau)\,\vartheta_{01}(\tau)\,\vartheta_{10}(\tau)=-2\eta^3(\tau)$. Hence $\varphi=2$ which is a trivial modular function for $\Gamma[1]$.

ii) $N=2$, $n=8$, $a=(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0)$, $b=(0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$.

Then the theta function is $\vartheta_{01}^4\vartheta_{10}^4\in\{\Gamma_\vartheta,\,-4,v\}$.

The relation we find is

$$-\vartheta_{01}^4\vartheta_{10}^4+\vartheta_{00}^4\vartheta_{10}^4+\vartheta_{00}^4\vartheta_{01}^4=G_4. \tag{4.3}$$

This can be derived from (4.1).

iii) $N=2$, $n=15$, $a=\frac{1}{2}(0, 1, \dots 1)$, $b=\frac{1}{2}(1, 1, 1, 1, 1, 1, 0, \dots 0)$.

The theta function is $-\vartheta_{10}\,\vartheta_{00}^5\,\vartheta_{01}^9 \in \left\{\Gamma[2], -\frac{15}{2}, v\right\}$. In this case there are two characters of $\Gamma[1]$ which are equal to the multiplier of φ on $\Gamma[2]$. Therefore we find two operators mapping φ into a modular function of $\Gamma[1]$. The two relations can be written as follows:

$$\text{Define } F = -\vartheta_{00}^4\,\vartheta_{01}^8 + \vartheta_{10}^4\,\vartheta_{00}^8 + \vartheta_{01}^4\,\vartheta_{10}^8$$

$$G = \vartheta_{01}^4\,\vartheta_{00}^8 - \vartheta_{00}^4\,\vartheta_{10}^8 + \vartheta_{10}^4\,\vartheta_{01}^8.$$

Then F and G are modular forms of $\{\Gamma, -6, 1\}$ where Γ is the subgroup of index 2 of $\Gamma[1]$ generated by U^2 and UT.

We have

$$\left\{\begin{aligned} F+G &= 48\,\eta^{12} \\ F-G &= -2\,G_6. \end{aligned}\right. \tag{4.4}$$

iv) $N=3$, $n=6$, $a=\frac{1}{3}(0, 0, 0, 1, 1, 1)$, $b=\frac{1}{3}(1, 1, 1, 0, 0, 0)$.

Then $\vartheta\left(\tau\middle|\begin{matrix}a\\b\end{matrix}\right) = e^{\frac{\pi i}{3}}\sqrt{3}\,\eta^3(3\tau)\,\eta^3\!\left(\frac{\tau}{3}\right)$.

In this case φ is mapped into 0 by the operator and the relation we find can be written as $\eta^6|T(3)=0$.

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